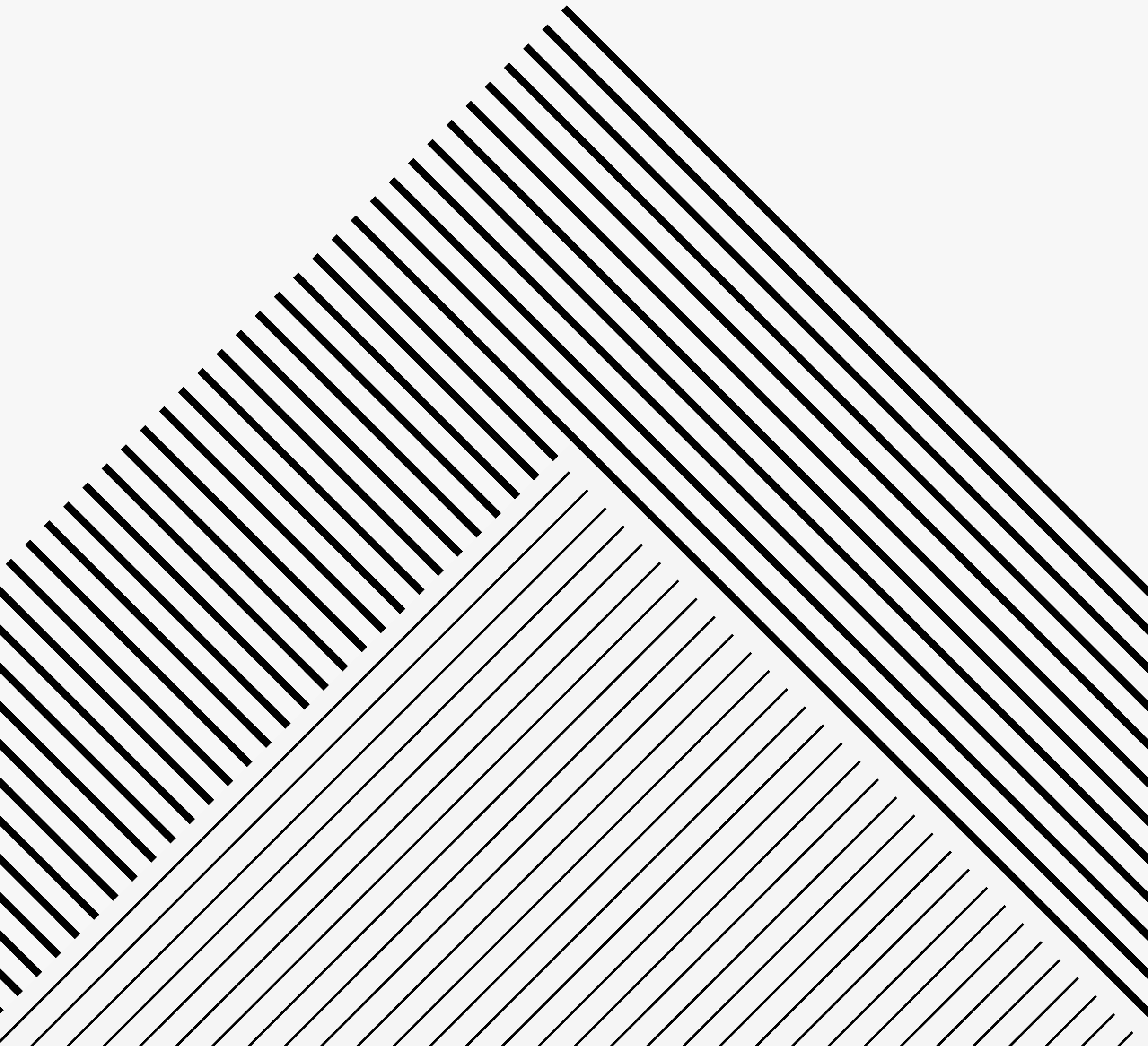
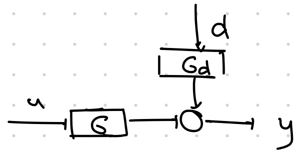


LECTURE 1: PRELIMINARIES

Introduction

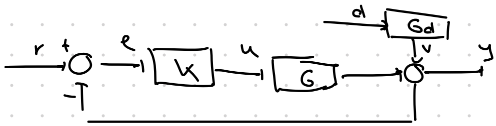


> **Regulator problem**: manipulate u to counteract d

> **Servo problem**: manipulate u to track r

↳ For both: minimize the control error

↳ We can add feedback and a controller K



↳ K is designed with info about d and/or r, G, and G_d

> We define a class of models: $G_p = \underbrace{G}_{\text{nominal plant}} + \underbrace{\Delta}_{\text{uncertainty}}$ with $\Delta \leq 1$

↳ **Nominal stability**: G is stable

↳ **Nominal performance**: G satisfies performance bounds

↳ **Robust stability**: G_p is stable for all systems in the class

↳ **Robust performance**: G_p satisfies performance bounds for systems in the class

> We work with rational LTI MIMO transfer functions $G(s)$

↳ If the system is proper we can have a state-space system

> **Scaling** is done for: ① numerical stability
② weighting (MIMO)
③ norm bounded control

↳ done by scaling $\frac{d}{d_{max}}$, $\frac{u}{u_{max}}$, and for e, r, y

> Actuator / sensor placement plays a role
> Linearization / working point plays a role
> Uncertainty in parameters plays a role

Frequency Response Function

> $G(s) \rightarrow G(j\omega)$ only positive imaginary axis

> FRF is a systems response to sinusoidal frequencies at ω

> Assume input $u = u_0 \sin(\omega t + \alpha)$

↳ output is then $y = y_0 \sin(\omega t + \beta)$

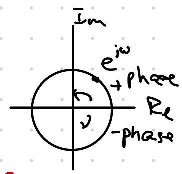
↳ $\| G(j\omega) \| = \frac{y_0}{u_0}$ ↳ $\angle G(j\omega) = \beta - \alpha$

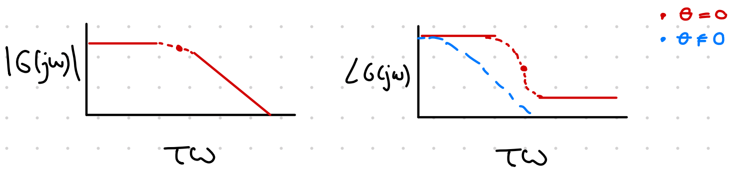
↳ Plotting these gives the Bode plot

EXAMPLE

$$G(s) = \frac{k e^{-\theta s}}{\tau s + 1} \rightarrow G(j\omega) = \frac{k e^{-\theta j \omega}}{j\omega\tau + 1}$$

$$\| G(j\omega) \| = \frac{|k| \cdot 1}{\sqrt{(\omega\tau)^2 + 1}}, \quad \angle G(j\omega) = -\theta\omega - \tan^{-1}\left(\frac{\omega\tau}{1}\right)$$





> Minimum phase systems: there exists a unique relation between the gain and phase of a FRF (for stable systems)

↳ All but systems with right half plane zeros or time delays

↳ $\angle G(j\omega) \approx \frac{\pi}{2} N(\omega_0)$

Feedback, Stability, and Performance

> The mapping from the disturbance to the output is the sensitivity function
> The mapping from the reference to the output is the complementary sensitivity function T

↳ $S + T = I$; $y = Sv + Tr$

> We can define the loop transfer function L:

↳ $y = \frac{1}{1+L} v + \frac{L}{1+L} r$; $L = GK$

> Even without feedback, we need the inverse of the plant, which might be hard, and there are signal and model uncertainties

> For high gain feedback ($L >> 1$), $u = G^{-1} r - G^{-1} G_d d$, similar as model inversion

↳ problem: stability

↳ check: ① All poles in LHP
② Nyquist
③ (OL, stable, MPS): Bode stability condition: $|L(j\omega_{180})| < 1$

> Performance: ① Rise time
② Overshoot
③ Damping
④ Settling time
} time domain

① Gain margin
② Phase margin
③ Max $\theta_d = \frac{PM}{\omega_c}$
Frequency domain

① ω_c from L — cross 0 dB line
② ω_{BT} from T — cross -3 dB line
③ ω_B from S — cross -3 dB line

① $M_S = \max_\omega |S(j\omega)| = \|S\|_\infty$
② $M_T = \max_\omega |T(j\omega)| = \|T\|_\infty$

$\|S\| - \|T\| \leq |S+T| = 1$

↳ $GM \geq \frac{M_S}{M_S - 1}$; $PM \geq 2\sin^{-1}\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$

↳ $GM \geq 1 + \frac{1}{M_T}$; $PM \geq 2\sin^{-1}\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T}$

Loop Shaping

> Approaches: ① Shape transfer functions: ⓐ L
ⓑ T, S, KS — we do these
② Signal based approach (LQG, H_2, H_∞)
③ MPC

> Key idea: shape L using Nyquist/Bode for closed loop perf/stab.

> L: ① Low freq: high gain
② High freq: low gain
③ Cross-over: good phase margin

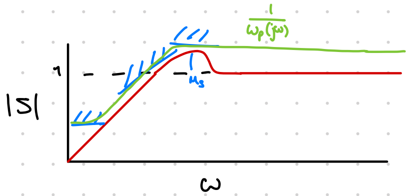
> S: ① Low freq: low gain → disturbance rejection
② High freq: gain ≈ 1 → no difference OL vs CL
③ cross-over: peak

> T: ① Low freq: gain $= 1$ → tracking
② High freq: low gain → no tracking
③ cross-over: peak

> Design process: ① Determine ω_{gc}
② LPF
③ Add phase
④ Set cross-over frequency
⑤ Reference tracking

> The H_∞ norm of a scalar and stable transfer function $f(s)$ is the peak value of $|f(j\omega)|$ as a function of frequency

- The sensitivity is a good performance indicator



$$S(j\omega) < \frac{1}{w_p(j\omega)} \quad \forall\, \omega \quad \rightarrow \text{Nominal performance}$$

$$\downarrow$$

$$\| w_p(s)\, S(s) \|_\infty < 1$$

$$\left\| \begin{matrix} w_u(s)\, K(s) \\ w_T(s)\, T(s) \end{matrix} \right\|$$

LECTURE 2: MIMO CONTROL

Introduction

> We consider $\vec{y} = G\vec{u}$

↳ One input can affect multiple outputs → both have a direction

- $GK \neq KG$

> Push through rule: $G_1(I - G_2G_1)^{-1} = (I - G_1G_2)^{-1}G_1$

> For multiple disturbances, we can have an output and input S and T

MIMO FRF

> The gain of a MIMO system equals $\dfrac{\|y(\omega)\|_2}{\|d(\omega)\|_2}$

↳ depends on direction of input

↳ Max, min $= \bar{\sigma}(G(j\omega)),\ \underline{\sigma}(G(j\omega))$

> Eigenvalues can not be used for non-square, and can be misleading
↳ thus, we use singular value decomposition.

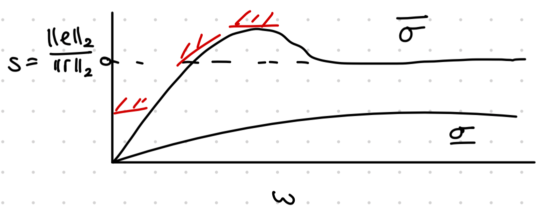
↳ $G(\omega_0) = \underbrace{U}_{\text{outp. dir.}}\Sigma\underbrace{V^H}_{\text{inp. dir.}}$

↳ V actually is I. Thus $GV = U\Sigma$

> We know S holds many properties. For MIMO: $\dfrac{\|e\|_2}{\|r\|_2}$, which depends on direction

↳ It can be bounded: $\underline{\sigma}(S(j\omega)) \leq \dfrac{\|e\|_2}{\|r\|_2} \leq \bar{\sigma}(S(j\omega))$

↳ For performance we want gain small in all directions and look at $\bar{\sigma}$



↳ Weighted sensitivity design: $\bar{\sigma} < \dfrac{1}{|w_p(j\omega)|}\ \forall\omega$; $\|w_pS\|_\infty < 1$

> The bandwidth ω_B is the ω where $\bar{\sigma}$ crosses 0.7 from below (at least ω_B)

> Condition number: $\gamma = \dfrac{\bar{\sigma}}{\underline{\sigma}}$

↳ large: ill-conditioned, control issues

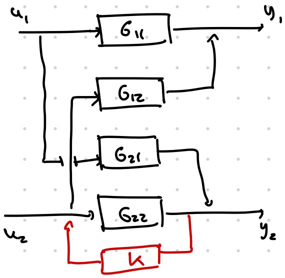
Relative Gain Array

> The RGA is: $RGA(G) = \Lambda(G) = G \times (G^{-1})^T$ (el. wise mult.)

↳ Measure of gain with no feedback vs perfect feedback $\quad \lambda_{11} = \dfrac{\left.\frac{\partial y_1}{\partial u_1}\right|_{OL}}{\left.\frac{\partial y_1}{\partial u_1}\right|_{CL}}$

↳ RGA is frequency dependent

EXAMPLE



OL: $u_2 = 0$

$$\left.\frac{\partial y_1}{\partial u_1}\right|_{OL} = G_{11}$$

Perfect FB on channel 2: $y_2 = c$

↳ $c = G_{22} u_2 + G_{21} u_1 \Rightarrow u_2 = -G_{22}^{-1} G_{21} u_1$

$$\left.\frac{\partial y_1}{\partial u_1}\right|_{CL} = G_{11} - G_{12} G_{22}^{-1} G_{21}$$

↳ $\lambda_{11} = \dfrac{I}{I - \frac{G_{12} G_{21}}{G_{22} G_{11}}}$

↳ you want a ratio ≈ 1 as then the channels are decoupled

↳ Wanted is an RGA close to I

↳ A system can be rearranged to alter I/O pairings for this

MIMO Design

> Easiest MIMO is with diagonal controller (decentralized control)

↳ A system can be decoupled: $\hat{G}(s) = G(s) W_1$

↳ Options: ① $\hat{G} = G G^{-1}$; $K = \frac{k}{s} I$

② $\hat{G}(s) = G(s) G(0)^{-1}$

③ $\hat{G}(s) = G(s) G(j\omega_0)^{-1}$

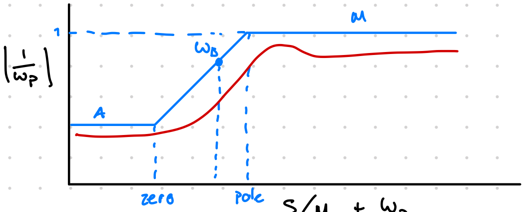
④ $\hat{G}(s) = U(j\omega_0)^T G(s) V(j\omega_0) = W_2 G(s) W_1$

> other method: mixed sensitivity

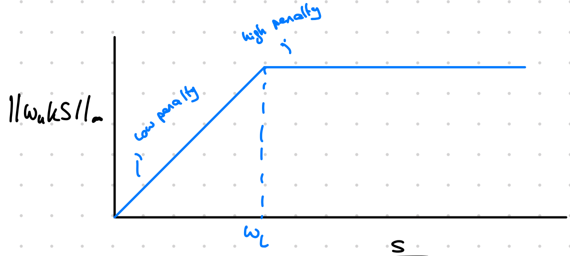
↳ $\|N\|_\infty = \max_\omega \bar{\sigma}(N(j\omega)) < 1$; $N = \begin{bmatrix} w_p S \\ w_u K S \end{bmatrix}$

↳ w_p typically $\mathrm{diag}(w_{p,i})$

↳ low common I or $\frac{s}{s+\omega_1}$



$$w_p = \frac{s/M + \omega_B}{s + \omega_B A}$$



$$w_u = \frac{s}{s+\omega_L}$$

> In MIMO, a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(z_i)$

EXAMPLE

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

$$\det(G(s)) = \frac{1}{(s+2)^2}\cdot\left[(s-1)(2(s-1)) - 18\right]$$

$$= \frac{1}{(s+2)^2}\cdot\left[2s^2-4s-16\right]$$

$$= 2\frac{s-4}{s+2} \longrightarrow \text{zero at } 4, \text{ pole at } -2$$

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix} \rightarrow \text{No zeros}$$

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \\ \frac{s-1}{s+2} & \frac{s-2}{s+1} \end{bmatrix} \rightarrow s=1 \text{ or } s=2 \quad (\text{lin. dep. col})$$

↳ Also: $s = -\frac{3}{2}$

MIMO Robustness

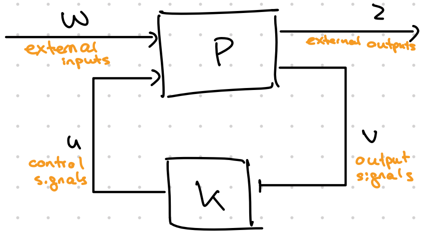
> GM and PM do not provide RS information. Large $\bar{\sigma}(S)$ indicate issues

> High RGA values are a reason of concern

> RP can be seen from the response

↳ RGA and $\bar{\sigma}(S)$ are indicators for robustness, but we need better tools for analysis and synthesis

Generalized Plant



> We want to find a K which counteracts the influence of w on z

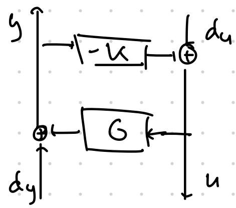
> We can embed mixed sensitivity in this framework

$$z = \underbrace{\begin{bmatrix} W_p S \\ W_u K S \\ W_T T \end{bmatrix}}_{N} w \longrightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} W_p & W_p G \\ 0 & -W_u \\ 0 & -W_T G \\ -I & -G \end{bmatrix}}_{P} \begin{bmatrix} w \\ u \end{bmatrix}$$

↳ General: $\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$

LECTURE 3: LINEAR SYSTEM THEORY & PERF. LIMITATIONS

Internal Stability & Q-parameterization

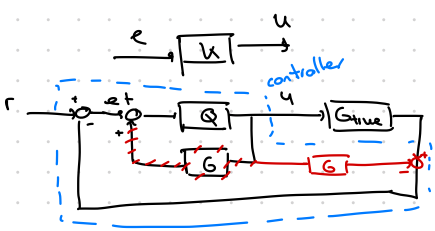


- Internal stability I: Assume G and K contain no unstable hidden modes. IS if all TF are stable
- Internal stability II: All RHP poles are contained in the min. realization of GK and KG. IS if either TF stable
- $u = (I+KG)^{-1} d_u - K(I+GK)^{-1} d_y$
- $y = G(I+KG)^{-1} d_u + (I+GK)^{-1} d_y$

- If we have IS: ① If $G(s)$ has a RHP zero at z, so do L, L_I, T, SG, T_I
  ② If $G(s)$ has a RHP pole at p, so do L, L_I, and a RHP zero for S, KS, S_I → canceling of pole/zero

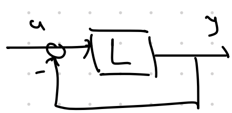
- Internal Model Control refactors the TF. Given $Q = K(I+GK)^{-1}$

  ↳ $u = (I - QG) d_u - Q d_y$
  ↳ $y = (G - GQG) d_u + (I - GQ) d_y$
  ↳ IS if plant and Q are stable
  ↳ Idea is that the controller has a copy of the plant
  ↳ $K = (I - QG)^{-1} Q$



Stability Analysis

- Stability: no RHP poles
- Bode stability criteria only for SISO
- We can extend Nyquist



$$L(s) = \left[\begin{array}{cc} A_{OL} & B_{OL} \\ C_{OL} & D_{OL} \end{array}\right] = C_{OL}(sI - A_{OL})^{-1} B_{OL} + D_{OL}$$

↳ $\det(sI - A_{OL}) = \phi_{OL}$ → zeros are OL poles

①

> We can derive : $\dot{x} = (A_{OL} - B_{OL}(I + D_{OL})^{-1} C_{OL}) x = A_{CL} x$

$$\phi_{CL} = \det(sI - A_{CL}) \quad ②$$

③ $\det(I+L) = \det(I + C_{OL}(sI - A_{OL})^{-1} B_{OL} + D_{OL})$ ④ constant

> Using the Schur's complement : $\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12})$
$= \det(A_{22}) \det(A_{11} - A_{12} A_{22}^{-1} A_{21})$

↳ we set : $A_{11} = I + D_{OL}$
$A_{21} = -B_{OL}$
$A_{22} = sI - A_{OL}$
$A_{12} = C_{OL}$

↳ Then : $\underbrace{\det(I + D_{OL})}_{④} \underbrace{\det(sI - A_{CL} + B_{OL}(I + D_{OL})^{-1} C_{OL})}_{②}$

$$\underbrace{\det(sI - A_{OL})}_{①} \underbrace{\det(I + D_{OL} + C_{OL}(sI - A_{OL})^{-1} B_{OL})}_{③}$$

↳ $\boxed{\det(I + L) = \frac{c \cdot \phi_{CL}}{\phi_{OL}}}$

↳ Cauchy : Evaluate a TF $F(s)$ along a cw contour C, the map encircles the origin $N = \underbrace{Z}_{\text{\# zeros in C}} - \underbrace{P}_{\text{\# poles in C}}$ times cw

↳ Nyquist : C is the RHP Then, apply the above equation.

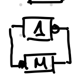
↳ Generalized Nyquist : N is the number cw encirclements of origin by contour map $\det(I + L(s))$

> The small gain theorem : Consider a system with stable $L(s)$, then the CL is stable if $\|L(j\omega)\| < 1 \; \forall \omega$,

System Norms

> we had : $\|L\| < 1 \rightarrow$ stable

↳ Now let $L = M\Delta$, $\|\Delta\| \leq 1$



↳ Small gain theorem : $\|M\Delta\| < 1 \rightarrow$ stable

↳ we want to write like : $\|M\Delta\| \leq \|M\| \|\Delta\| < 1$

↳ This multiplicative property is true for the ∞-norm

> Norms give a single number of vector, matrix, ...

↳ Vector : $\|a\|_p = (\sum_i |a_i|^p)^{\frac{1}{p}}$, Matrix : $\|A\|_F = \sqrt{\operatorname{tr}(A^H A)}$

> For a generalized plant : $\min_K \|N\| \rightarrow$ which norm ∞

> We have:

$$w \longrightarrow \boxed{G} \longrightarrow z$$

↳ with w input, z output, G stable TF with $g(t)$. Given w, how big can z become?

↳ Output: $\|z(t)\|_2 = \sqrt{\sum_i \int_0^\infty |z_i(t)|^2 d\tau}$

↳ Inputs: ① $w(t)$ is series of unit impulses $\mathcal{H}_2$
② $w(t)$ any signal $\|w(t)\|=1$ $\mathcal{H}_\infty$
③ $w(t)$ any signal $\|w(t)\|=1$, but $w(t)=0$ for $t\geq 0$ measure z for $t \geq 0$ Hankel

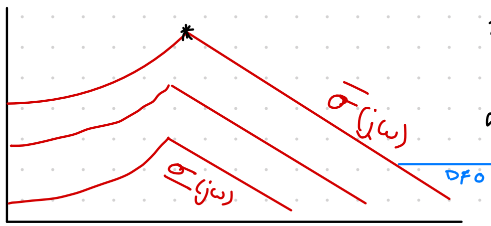
> Assuming strictly proper ($D=0$), for $\mathcal{H}_2$, $w(t)$ is a series of impulses

↳ $\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{tr\left(G(j\omega)^H G\right)}_{\|G(j\omega)\|_F^2 = \sum \sigma_i^2(G(j\omega))} d\omega}$; $\|G(s)\|_2 = \|g(t)\|_2 = \sqrt{\int_0^\infty tr\left(g(t)^H g(t)\right) dt}$

↳ Frequency domain ↳ Time domain

> Assuming proper ($D \neq 0$), for $\mathcal{H}_\infty$, $\|w(t)\|_2 = 1$:

↳ $\|G(s)\|_\infty = \max_\omega \bar{\sigma}\left(G(j\omega)\right)$; $\|G(s)\| = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} = \max_{\|w(t)\|_2=1} \|z(t)\|_2$



2-norm: square all values, and add

∞-norm: peak *

→ 2-norm → ∞ → not possible

↳ ∞-norm minimizes the peak (worst case scenario), 2-norm minimizes all singular values

↳ ∞-norm has the multiplicative property, which is why we use it

↳ Considering 2 systems in series: $w \longrightarrow \boxed{G_1} \xrightarrow{v} \boxed{G_2} \longrightarrow z$

↳ $\|G_2 G_1\|_\infty = \max_{w\neq 0} \frac{\|G_2 G_1 w\|_2}{\|w\|_2} \leq \max_{v \neq 0} \frac{\|G_2 v\|_2}{\|v\|_2} \cdot \max_{w \neq 0} \frac{\|G_1 w\|_2}{\|w\|_2}$

↳ $\frac{\|G_2 G_1 w\|_2}{\|w\|_2} \cdot \frac{\|v\|}{\|v\|} = \frac{\|G_2 v\|}{\|v\|} \cdot \frac{\|G_1 w\|}{\|w\|}$

↳ We can thus independently optimize v and w (better than just w)

Fundamental Limitations on Sensitivity

① $S + T = I$; $S = (I+L)^{-1}$; $T = L(I+L)^{-1}$

↳ For every ω either $|T|$ or $|S| > 0.5$, and $||T|-|S|| < 1$

② Interpolation constraints

① If p is a RHP pole of L, then $T(p)=1$, $S(p)=0$ $LS=T$ ↳ $L(p) \to$ $S(p)=0$

② If z is a RHP zero of L, then $T(z)=0$, $S(z)=1$

③ Waterbed effect

↳ Assume L has a relative degree ≥ 2, and N_p RHP poles. Then:

↳ $\int_0^\infty \ln|S(j\omega)|\,d\omega = \pi \cdot \sum_{N_p} Re(p_i)$

↳ Assume L has a single RHP zero at z. Then:

↳ $\int_0^\infty \ln|S(j\omega)|\,w(z,\omega)\,d\omega = \pi \ln \prod_{N_p} \left|\frac{p_i+z}{\bar{p}_i-z}\right|$; $w(z,\omega) = \frac{2z}{z^2+\omega^2}$

④ Time delays ($e^{-\theta s}$) limit the bandwidth to less than $\frac{1}{\theta}$

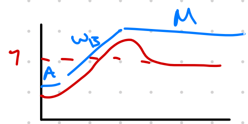
↳ Pade approximation: $e^{-\theta s} \simeq \frac{(1-\theta/2n\, s)^n}{(1+\theta/2n\, s)^n}$ cause a RHP zeros !

⑤ System zeros are invariant under feedback, and poles go to zeros

EXAMPLE

Consider $w_p(s) = \frac{s/M + \omega_B}{s + \omega_B A}$

$\|w_p S\|_\infty < 1$



Maximum modulus principle:

$\|f(s)\|_\infty = \max_\omega |f(j\omega)| \geq |f(s_0)|$ $\forall$ s_0 RHP zeros

Assume z in RHP is real. Then:

$|w_p(z)S(z)| \leq \|w_p S\|_\infty < 1$. Since $S(z)=1$:

$|w_p(z)| < 1$. For $A=0$, $M=2$: $\frac{z/2 + \omega_B}{z} = w_p(z) < 1$

↳ $\omega_B < \frac{z}{2}$ → RHP zeros limit the bandwidth

> All 5 also hold for MIMO, but we have to think about directions

LECTURE 4: SISO UNCERTAINTY AND ROBUSTNESS

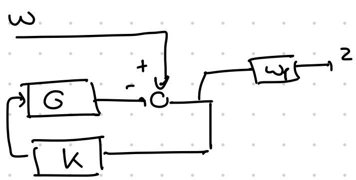
Introduction to Robustness

> A system is robust if it is insensitive to differences between the actual system and the model used for controller synthesis

↳ The difference is the uncertainty

↳ In H_∞, we check RS and RP for the worst-case scenario

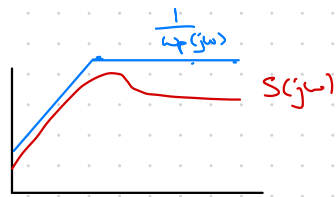
> we have in this lecture: $N = w_p S$ SISO, generalized plant



$z = w_p S w = N w$

↳ NS : iff poles of N are in LHP

↳ NP : $\|w_p S\|_\infty < 1$ or $|w_p(j\omega) S(j\omega)| < 1 \;\forall\, \omega$ →



↳ RS : $\|w_I T\|_\infty < 1$ or $|w_I(j\omega) T(j\omega)| < 1 \;\forall\, \omega$

↳ RP : $\max\limits_{\omega} |w_p(j\omega)S(j\omega)| + |w_I(j\omega)T(j\omega)| < 1$

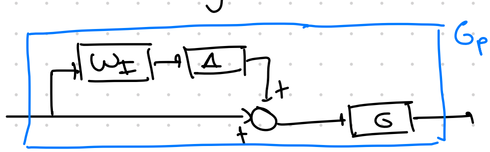
> Approach: ① Determine the uncertainty set
② Check for RS in uncertainty set
③ Check for RP in uncertainty set

① $\Pi = G + G w_I \Delta$, where w_I is used for scaling, and $\|\Delta\|_\infty < 1$ are all normalized perturbations

Representing Uncertainty

> uncertainty comes from: ① unknown model parameters
② Nonlinearity
③ Imperfections in measurements
④ Inaccuracies for high frequencies
⑤ Low order model is preferable

↳ 2 classes : ① Parametric : assume a parameter is bounded
② Dynamic : normalize complex perturbations



> We often lump dynamic uncertainties into multiplicative:

↳ Π_I : $G_p(s) = G(s)(1 + w_I(s)\Delta_I(s))$; $|\Delta_I(j\omega)| \leq 1 \ \forall \omega$

> Parametric gain uncertainties: suppose $G_p = k_p G_0$;

↳ $G_p = \bar{k} G_0 (1 + r_k \Delta)$; $|\Delta| < 1$

> Time constant uncertainties: suppose $G_p = \frac{1}{\tau_p s + 1} G_0$:

↳ $G_p = \frac{G_0}{1 + \bar{\tau} s} \cdot \frac{1}{1 + w_{i1}\Delta}$; $w_{i1} = \frac{r_\tau \bar{\tau} s}{1 + \bar{\tau} s}$

> Pole uncertainty: same as time constant

EXAMPLE

Consider $\dot{x} = \begin{bmatrix} -(1+k) & 0 \\ 1 & -(1+k) \end{bmatrix} x + \begin{bmatrix} 1-k/k \\ -1 \end{bmatrix} u$

$y = [1 \ \ \alpha] x$

with $k = 0.5 + 0.1\delta_1$ and $\alpha = 1 + 0.2\delta_2$

In MATLAB we can just plug this in

Representing Uncertainty in the Frequency Domain

> Parametric uncertainty is usually replaced by complex uncertainty,

↳ $|\Delta| \leq 1 \longrightarrow |\Delta(j\omega)| \leq 1$

> Additive uncertainty:

↳ Π_A : $G_p = G + w_A \Delta_A$; $|\Delta_A(j\omega)| \leq 1 \ \forall \omega$

> Multiplicative uncertainty:

↳ Π_I : $G_p = G(1 + w_I \Delta_I)$; $\Delta_I(j\omega) \leq 1 \ \forall \omega$

↳ with $|w_I(j\omega)| = \frac{|w_A(j\omega)|}{|G(j\omega)|}$, these are equal

> To construct Δ_A and Δ_I: ① Select nominal model G
② ⓐ Additive: find smallest radius including all possible plants, and then fit
ⓑ Multiplicative: find smallest l_I and fit

> Choice of nominal model: ① Mean parameter values
② Simplified model
③ Smallest discs

> We can consider $G_p = G_0 f$, with f neglected dynamics

> For unmodelled dynamics, we typically use a set w_I multiplicative

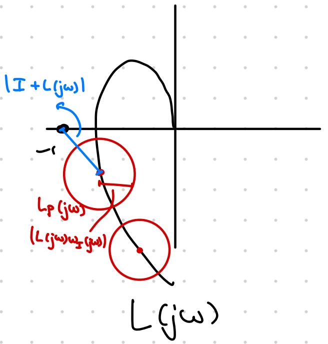
Robust Stability

> Adding a negative feedback controller to the previous diagram:

↳ $L_p = G_pK = GK(1 + w_I\Delta_I) = L + w_I L\Delta_I$, with $|\Delta_I(j\omega)| \leq 1 \ \forall \omega$

↳ How do we check for RS?

> Assume G is stable and NS. Then the Nyquist does not encircle -1



$$\Rightarrow |I + L(j\omega)| > |L(j\omega) w_I(j\omega)| \quad \forall \omega$$

↳ $1 > \dfrac{|L(j\omega) w_I(j\omega)|}{|I + L(j\omega)|} \quad \forall \omega$

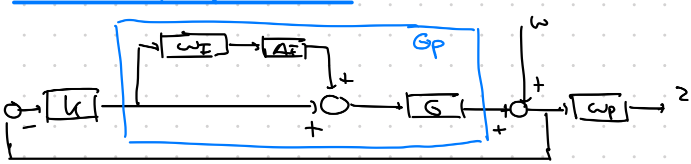
↳ $|w_I(j\omega) T(j\omega)| < 1 \quad \forall \omega$

↳ $\|w_I T\|_\infty < 1$

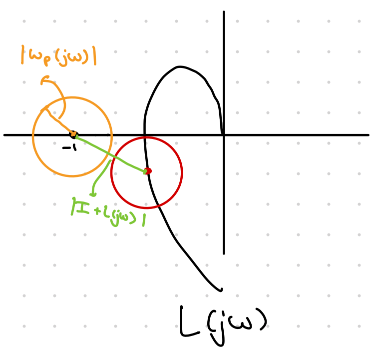
pretty much O can not contain -1

> Replacing L by $M\Delta$: $|M(j\omega)| < 1 \ \forall \omega$

Robust Performance



> From NP: $|w_p(j\omega)| < |I + L(j\omega)|$



$$\Rightarrow |I + L(j\omega)| > |w_p(j\omega)| + |L(j\omega) w_I(j\omega)|$$

↳ $|w_p(j\omega) S(j\omega)| + |w_I(j\omega) T(j\omega)| < 1 \quad \forall \omega$

↳ RP condition

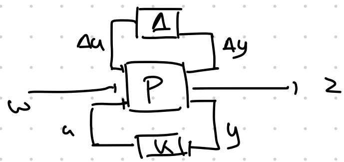
O and O can not overlap

LECTURE 5: MIMO ROBUST STABILITY

General Control Configuration

> For MIMO, we will have multiple Δ_i

↳ Approach: collect all and put them in a diagonal matrix



> For every Δ_i we have $\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall \omega$
> Because Δ is diagonal, $\|\Delta\|_\infty \leq 1$
> The Δ has structure, so not all Δ's are allowed (diagonal)

↳ Thus, we need the structured singular value μ for analysis and synthesis

Representing Uncertainty for MIMO

> Parametric uncertainties from SISO carry over to MIMO

> Additive Π_A : $G_p = G + w_A \Delta_A$
> Multiplicative Π_I : $G_p = G(I + w_I \Delta_I)$ or output
> Inverse

> For SISO, $l_o = l_I$, for MIMO: $l_o \sim \gamma l_I$, γ the condition number

> Input uncertainties:

> Physical input : $m_i = h_i(s) u_i$

> Unknown dynamics is absorbed in the model

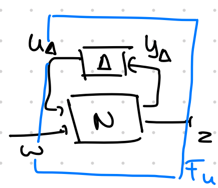
↳ $h_{p_i}(s) = h_i(s)(1 + w_{I_i}(s)\delta_i(s))$; $|\delta_i(j\omega)| \leq 1 \ \forall \ \omega$

↳ $G_p = G(I + w_I \Delta_I)$, w_I and Δ_I diagonal → No shared uncertainty

Robust Stability and Performance

> We have the $N\Delta$ structure :

↳ $\begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix}$



↳ $F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1} N_{12}$

↳ NS : N is internally stable

↳ NP: $\|N_{22}\|_\infty < 1$ and NS

↳ RS: $F_u(N,\Delta)$ stable for all Δ, $\|\Delta\|_\infty \le 1$ and NS

↳ RP: $\|F_u(N,\Delta)\|_\infty < 1$ for all Δ, $\|\Delta\|_\infty < 1$ and NS

↳ Assuming Δ is stable, only $(I-N_{11}\Delta)^{-1}$ can make F_u unstable

> Generalized Nyquist: $\det(I-M\Delta)$ does not encircle 0 $\forall \Delta$

↳ $\det(I-M\Delta) \neq$ $\forall \Delta\ \forall \omega$

↳ $\lambda_i(M\Delta) \neq 1\ \forall i\ \forall \Delta\ \forall \omega$, but Δ can allow scaling, thus:

↳ RS: $\rho(M\Delta(j\omega)) < 1\ \forall \omega\ \forall \Delta$ (or: $\max_\Delta \rho(M\Delta(j\omega)) < 1\ \forall \omega$

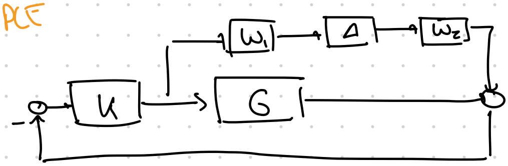
↳ spectral radius ?

> Now assume that Δ is allowed to be any complex TF with $\|\Delta\|_\infty \le 1$

↳ $\max_\Delta \rho(M\Delta) = \max_\Delta \bar{\sigma}(M\Delta) = \max_\Delta \bar{\sigma}(\Delta)\bar{\sigma}(M) = \bar{\sigma}(M)$

↳ $\bar{\sigma}(M) < 1\ \forall \omega$ (or $\|M\|_\infty < 1$)

EXAMPLE



$M = W_1 K (I + GK)^{-1} W_2$

↳ $\|M\|_\infty < 1$ for RS

> Let $E = W_1 \Delta W_2$. Let $M = W_1 M_0 W_2$. We get:

↳ ① $G_p = G + E_A$: $M_0 = KS$
② $G_p = G(I + E_I)$: $M_0 = T_I$
③ $G_p = (I + E_O)G$: $M_0 = T$

↳ For all, RS: $\|W_1 M_0 W_2\|_\infty < 1$

> For a structured uncertainty: RS if $\bar{\sigma}(M) < 1\ \forall \omega$

> Scaling keeps the same Δ, but changes M to get $\|M\|_\infty < 1$

Structured Singular Value

> The structured singular value is a generalization of the maximum singular value

> We want to find the smallest structured Δ which makes $I - M\Delta$ singular then, $\mu(M) = 1/\bar{\sigma}(M)$

↳ $\frac{1}{\mu(M)} = \min\{k_m \mid \det(I - k_m M \Delta) = 0$ for structured $\bar{\sigma}(\Delta) \le 1\}$

↳ small μ good, big μ bad

↳ μ is a measure of how far away you are from stability

↳ μ depends on M and the structure of Δ

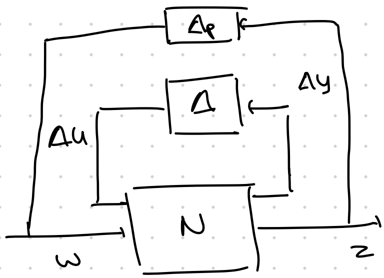
↳ RS iff $\mu(M(j\omega)) < 1 \ \forall \omega$

> Properties:
1. $\mu(M) = \max_{\Delta, \bar{\sigma}(\Delta) \leq 1} \rho(M\Delta)$
2. For a scalar α, $\alpha\mu(M) = \mu(\alpha M)$
3. For $\Delta = I\delta$, $\mu(M) = \rho(M)$
4. For a full complex Δ, $\mu(M) = \bar{\sigma}(M)$
5. In general, $\rho(M) \leq \mu(M) \leq \bar{\sigma}(M)$
6. For $\Delta D = D\Delta$, $\mu(DMD^{-1}) = \mu(M)$
7. Upper bound: $\mu(M) \leq \bar{\sigma}(DMD^{-1})$

LECTURE 6: ROBUST CONTROLLER SYNTHESIS

Robust Performance

> We have for SISO: ① RS: $|w_I T| < 1 \quad \forall \omega$
> ② RP: $|w_I T| + |w_P S| < 1 \quad \forall \omega$



$$\begin{bmatrix} \Delta y \\ z \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \Delta u \\ \omega \end{bmatrix}$$

NS: Check internal stability of N

NP: $\|N_{22}\|_\infty < 1$ or $\bar{\sigma}(N_{22}(j\omega)) < 1 \ \forall \omega$ or $\mu_{\Delta P}(N_{22}(j\omega)) < 1 \ \forall \omega$

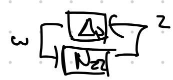
RS: $\mu_\Delta(N_{11}(j\omega)) < 1 \ \forall \omega$

RP: $\mu_{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}}(N_{11}(j\omega)) < 1 \ \forall \omega$

> With the above block diagram, we can rewrite the RP problem (without Δ_P) to a RS problem (with Δ_P) with structured uncertainty

> In H_∞, we have NP if $\|N_{22}\|_\infty < 1$

> Consider N_{22} with a Δ_P, $\|\Delta_P\|_\infty \le 1$



↳ Generalized Nyquist: $\det(I - N_{22}(j\omega)\Delta_P(j\omega)) = 0 \quad \forall \Delta_P, \forall \omega$

↳ We find the same RS $M\Delta$ structure

↳ Thus NP can be reformulated as RS

↳ Thus, we can combine NP and RS to look at RP

↳ RS: $\mu_{\hat{\Delta}}(N) < 1 \quad \forall \omega$; $\hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$, and NS Δ_P always full

D-K Iterations

> For MIMO, we check for NS, NP, RS, RP using μ-analysis:

↳ Seek a controller that minimizes a certain μ-condition: μ-synthesis

↳ There is no direct method, but we have an upper bound

↳ $\mu(N(K)) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DN(K)D^{-1})$

↳ We thus seek a controller for $\min_K \left(\min_{D \in \mathcal{D}} \|DN(K)D^{-1}\|_\infty \right)$

↳ We minimize $\|DN(K)D^{-1}\|_\infty$ by alternating D and K

> D-K iterations: ① K-step: controller that minimizes the scaled problem

↳ $\min_K \|DN(K)D^{-1}\|_\infty$

② D-Step : Find $D(j\omega)$ to minimize $\bar{\sigma}$ at all ω

↳ start with $D = I$

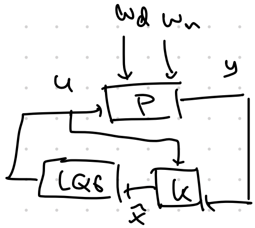
③ Fit a TF on top of $D(j\omega)$ and return to step ①

↳ D becomes part of the generalized plant

LECTURE 7: OVERVIEW

Trade-Offs in MIMO Feedback Design

> LQG controller:

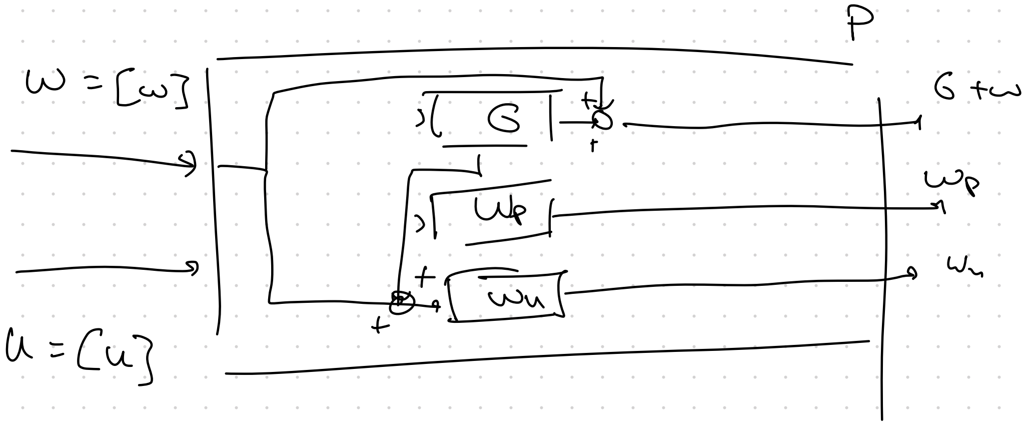


P is plant
K is Kalman filter

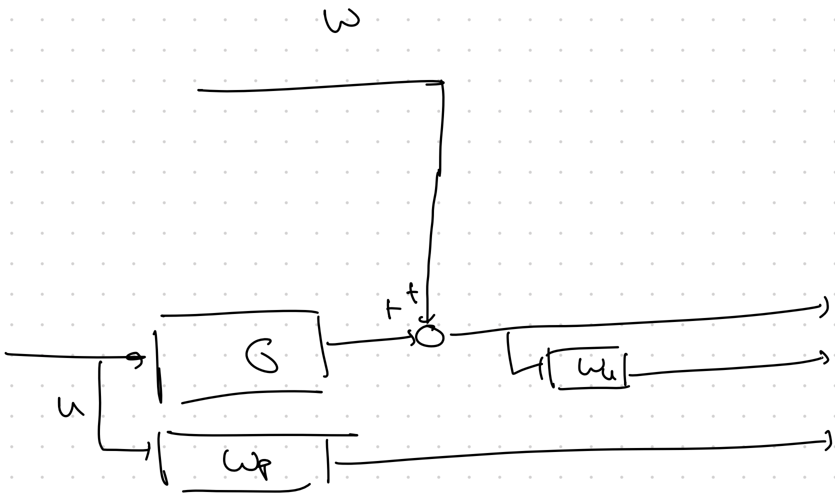
↳ No guaranteed margins → development for H_∞ control

$$\Delta y_i \longrightarrow \boxed{\Delta_i} \longrightarrow \Delta u_i$$

$$\Delta y_o \longrightarrow \boxed{\Delta_o} \longrightarrow \Delta u_o$$



P
w = [w]
u = [u]
G
Wp
Wu
G + w
Wp
Wu

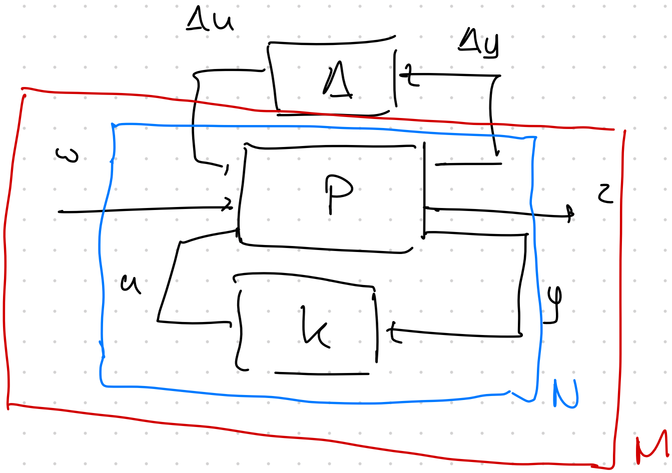


w
G
u
Wu
Wp

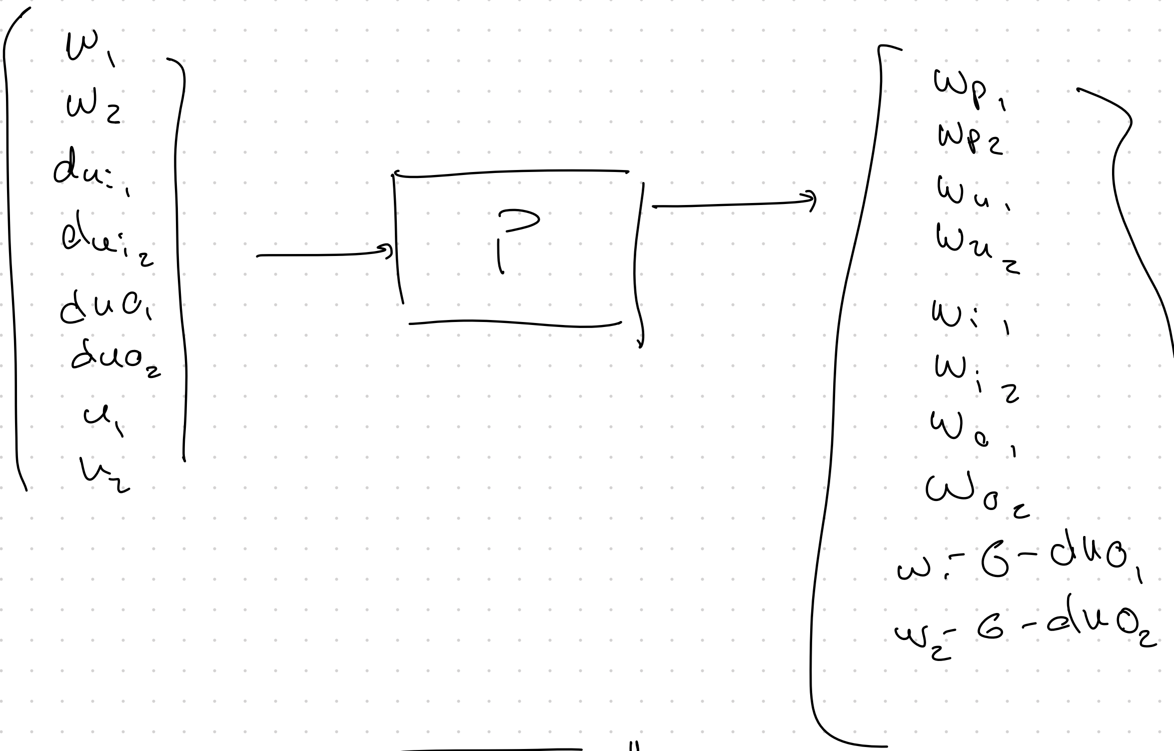
Δ is a block of all uncertainties

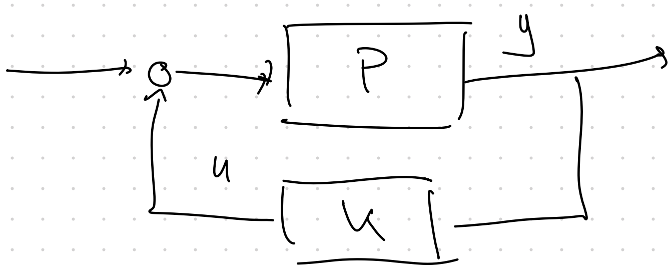
↳ For us: 4×4, diagonal, 4 dof.

Then:



$$N = \mathrm{lft}(P, K)$$
$$M = \mathrm{lft}(\Delta, N)$$





$$P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P(:,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P(1,:) = \begin{bmatrix} 1 \end{bmatrix}$$

$$P_p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$P_p - P(\text{cols}, \text{rows})$

$$P_p = P([2\ 1], :)$$

$$\text{out} = P\ \text{in}$$

$$\begin{bmatrix} \\ \\ \end{bmatrix} = P \begin{bmatrix} \\ \\ \end{bmatrix}$$

→ col = in
row = out

$$P_p = P(\text{out}, \text{in})$$

P

K